Eur. Phys. J. C 51, 207–215 (2007)
DOI 10.1140/epjc/s10052-007-0280-0

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

# Gauge generators, transformations and identities on a noncommutative space

R. Banerjee[a], S. Samanta[b]

S. N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India

Received: 8 February 2006 / Revised version: 2 March 2006 /
Published online: 5 April 2007 – 

**Abstract.** By abstracting a connection between gauge symmetry and gauge identity on a noncommutative space, we analyse star (deformed) gauge transformations with the usual Leibniz rule as well as undeformed gauge transformations with a twisted Leibniz rule. Explicit structures of the gauge generators in either case are computed. It is shown that, in the former case, the relation mapping the generator with the gauge identity is a star deformation of the commutative space result. In the latter case, on the other hand, this relation gets twisted to yield the desired map.

## 1 Introduction

Recent analysis [1–6] of gauge transformations in noncommutative theories reveals that, in extending gauge symmetries to a noncommutative space-time, there are two distinct possibilities. Gauge transformations are either deformed such that the standard comultiplication (Leibniz) rule holds or one retains the unmodified gauge transformations as in the commutative case at the expense of altering the Leibniz rule. In the latter case the new rule to compute the gauge variation of the star products of fields results from a twisted Hopf algebra of the universal enveloping algebra of the Lie algebra of the gauge group extended by translations.

While both these approaches preserve gauge invariance of the action, there is an important distinction. In the case of deforming gauge transformations into star gauge transformations, gauge symmetries act only on the fields in a similar way as in theories on commutative space-time. Star gauge symmetry can thus be interpreted as a physical symmetry in the usual sense. On the contrary, if ordinary gauge transformations are retained and a twisted Leibniz rule is implemented, then the transformations do not act only on the fields. Consequently it is not a physical symmetry in the conventional sense, and its connection with the previous case also remains obscure [5].

In this paper we analyse both these approaches within a common framework, which also illuminates a correspondence with the treatment of gauge symmetry in commutative space-time. To do this we recall that there is a general method of discussing gauge symmetries, either in the Lagrangian or Hamiltonian formulations, for theories on commutative space-time. We shall here concentrate on the Lagrangian version; for a detailed exposition, see [7]. It is known that, corresponding to each gauge symmetry, there is a gauge identity that is expressed in terms of the Euler derivatives. Moreover, this identity also involves the generator of infinitesimal gauge transformations in a very specific manner.

We extend this analysis to noncommutative gauge theories. A relation between the gauge generator and the gauge identity is derived. It is found to be a star deformation of the relation found in the usual commutative picture. From this relation and from knowledge of the gauge identity (obtained by simple inspection) the explicit form of the gauge generator is derived. Then the other viewpoint of keeping the gauge transformation undeformed at the price of a twisted Leibniz rule is considered. The generator of the undeformed gauge transformation is derived. Its structure is shown to be similar to the commutative space expression. Furthermore, we find that the relation connecting the gauge generator with the gauge identity (which is form invariant, irrespective of whether star or twisted gauge transformations are being considered) is neither the undeformed result nor its star deformation, as obtained in the previous treatment. Rather, it is a twisted form of the conventional (undeformed) result.

This paper is organised as follows. In Sect. 2 we briefly review the situation in which both interpretations of noncommutative gauge symmetry lead to identical conservation laws. Sections 3 and 4 give a detailed account of the computations for deformed gauge symmetry with the standard Leibniz rule and undeformed gauge symmetry with a twisted Leibniz rule, respectively. Explicit expressions for the generators and their connection with the gauge identity are analysed. Section 5 is a summary.

[a] e-mail: rabin@bose.res.in
[b] e-mail: saurav@bose.res.in

## 2 Gauge transformations and conservation law

Consider a theory on noncommutative space-time whose dynamics is governed by the action[1]

$$S = \int \mathrm{d}^4x \left[ -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}(x) * F^{\mu\nu}(x)) + \bar{\psi}(x) * (\mathrm{i}\gamma^\mu D_\mu * - m)\psi(x) \right], \quad (1)$$

where

$$D_\mu * \psi(x) \equiv \partial_\mu \psi(x) + \mathrm{i}g A_\mu(x) * \psi(x) \quad (2)$$

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \mathrm{i}g[A_\mu(x), A_\nu(x)]_* . \quad (3)$$

Here the star commutator is given by

$$[A_\mu(x), A_\nu(x)]_* = A_\mu(x) * A_\nu(x) - A_\nu(x) * A_\mu(x), \quad (4)$$

while the star product is defined as usually by

$$(f * g)(x) = \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu\right) f(x)g(y)\Big|_{x=y} \quad (5)$$

where $\theta^{\mu\nu}$ is a constant two index antisymmetric object.

The above action describes the noncommutative version of a non-Abelian theory that includes both the gauge and a matter (fermionic) sector with a proper interaction term. This action is invariant under both deformed gauge transformations,

$$\begin{aligned}
\delta_* A_\mu &= \mathcal{D}_\mu * \eta = \partial_\mu \eta + \mathrm{i}g(A_\mu * \eta - \eta * A_\mu)\\
\delta_* F_{\mu\nu} &= \mathrm{i}g[F_{\mu\nu}, \eta]_* = \mathrm{i}g(F_{\mu\nu} * \eta - \eta * F_{\mu\nu})\\
\delta_* \psi &= -\mathrm{i}g\eta * \psi\\
\delta_* \bar{\psi} &= \mathrm{i}g\bar{\psi} * \eta,
\end{aligned} \quad (6)$$

with the usual Leibniz rule,

$$\delta_*(f * g) = (\delta_* f) * g + f * (\delta_* g) \quad (7)$$

as well as the undeformed gauge transformations

$$\begin{aligned}
\delta_\eta A_\mu &= \mathcal{D}_\mu \eta = \partial_\mu \eta + \mathrm{i}g(A_\mu \eta - \eta A_\mu),\\
\delta_\eta F_{\mu\nu} &= \mathrm{i}g[F_{\mu\nu}, \eta] = \mathrm{i}g(F_{\mu\nu}\eta - \eta F_{\mu\nu})\\
\delta_\eta \psi &= -\mathrm{i}g\eta\psi\\
\delta_\eta \bar{\psi} &= \mathrm{i}g\bar{\psi}\eta,
\end{aligned} \quad (8)$$

with the twisted Leibniz rule [1–3],

$$\delta_\eta(f * g) = \sum_n \left(\frac{-\mathrm{i}}{2}\right)^n \frac{\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}}{n!} \times (\delta_{\partial_{\mu_1}\cdots\partial_{\mu_n}\eta} f * \partial_{\nu_1}\cdots\partial_{\nu_n} g + \partial_{\mu_1}\cdots\partial_{\mu_n} f * \delta_{\partial_{\nu_1}\cdots\partial_{\nu_n}\eta} g). \quad (9)$$

[1] We are considering a non-Abelian theory.

The essence of this modified rule is that the gauge parameter $\eta$ always remains outside the star operation, and hence is unaffected by it. This fact becomes important when we discuss the twisted gauge symmetry (8) and (9).

It is obvious from (3) and the definition of the gauge transformations (6) that, in general, both $A_\mu$ and $F_{\mu\nu}$ are enveloping algebra valued for deformed gauge symmetry. For the case of twisted gauge symmetry, (8) and (9), however, one has to consider the equation of motion derived later (see (23)), interpreted as equations for the gauge field $A_\mu$, to conclude that here also $A_\mu$ is enveloping algebra valued. The field tensor $F_{\mu\nu}$, by its very definition (3), is of course enveloping algebra valued. Thus, in both treatments of gauge symmetry, $A_\mu$ and $F_{\mu\nu}$ are enveloping algebra valued [2]. This implies that the gauge potential $A_\mu$ has to be expanded over a basis of the vector space spanned by the homogeneous polynomials in the generators of the Lie algebra,

$$A^\mu(x) = A^\mu_a(x) : T^a : + A^\mu_{a_1a_2}(x) : T^{a_1}T^{a_2} : + \cdots A^\mu_{a_1a_2\cdots a_n}(x) : T^{a_1}T^{a_2}\cdots T^{a_n} : + \cdots \quad (10)$$

where the double dots indicate totally symmetrised products,

$$\begin{aligned}
:T^a: &= T^a\\
:T^{a_1}T^{a_2}: &= \frac{1}{2}\{T^{a_1}, T^{a_2}\} = \frac{1}{2}(T^{a_1}T^{a_2} + T^{a_2}T^{a_1})\\
:T^{a_1}\cdots T^{a_n}: &= \frac{1}{n!}\sum_{\pi\in S_n} T^{a_{\pi(1)}}\cdots T^{a_{\pi(n)}} .
\end{aligned} \quad (11)$$

These symmetrised products may be simplified by using the basic Lie algebraic relation,

$$\left[T^a, T^b\right] = \mathrm{i}f^{abc}T^c, \quad (12)$$

where the $f^{abc}$ are the structure constants.

To be specific let us take the case of SU(2) in the two dimensional representation. In this representation the generators are the $2\times 2$ Pauli matrices ($T^a = \frac{\sigma^a}{2}$), satisfying (12), with $f^{abc} = \epsilon^{abc}$ and

$$\left\{\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right\} = \frac{1}{2}\delta^{ab} \quad (a = 1,2,3). \quad (13)$$

We may thus write $A_\mu$ as follows:

$$A_\mu = B_\mu + A^a_\mu \sigma^a \quad (a = 1,2,3). \quad (14)$$

It is also possible to interpret this situation as giving rise to the standard representation of U(2) with its four generators $(\mathbb{I}, \sigma^a)$:

$$A_\mu = A^a_\mu T^a \quad (T^a = \mathbb{I}, \sigma^a). \quad (15)$$

Similarly in the three dimensional representation of SU(2), the generators are defined in the adjoint representation, $(T^a)^{bc} = -\mathrm{i}\epsilon^{abc}$. Now the process of symmetrisation in (11) yields nine $3\times 3$ linearly independent hermitian matrices (these are the three $T^a$ and the six $\{T^a, T^b\}$), and hence we obtain the standard representation of U(3). This implies that the enveloping algebra valued $A_\mu$ given in (10) is equivalently simplified to a Lie algebraic U(3) representation with $A_\mu = A_\mu^a T^a$, where the $T^a$ are the nine generators of U(3).

In general one can verify that the generators given by (11), apart from forming a Lie algebra (12), also close under anticommutation [8, 9],

$$\{T^a, T^b\} = d^{abc} T^c \,. \tag{16}$$

The simpler nontrivial algebra that matches these conditions is U($N$) in the representation given by $N \times N$ hermitian matrices.

Following [10, 11], it is feasible to choose $T^1 = \frac{1}{\sqrt{2N}} \mathbb{I}_{(N\times N)}$ and the remaining $N^2 - 1$ of the $T$ as in SU($N$). Then the trace condition also follows:

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{17}$$

and $f^{abc}$ and $d^{abc}$ are completely antisymmetric and completely symmetric, respectively.

In the rest of this paper we will work with these simplifications. The gauge potential and the field strength will be explicitly written as,

$$A_\mu = A_\mu^a T^a \,, \tag{18}$$
$$F_{\mu\nu} = F_{\mu\nu}^a T^a \,, \tag{19}$$

where the $T^a$ are the $N^2$ hermitian generators of U($N$) that satisfy the conditions (12), (16) and (17).

In order to derive the field equations we need the following well known properties of the star product within an integral:

$$\int \mathrm{d}x A(x) * B(x) = \int \mathrm{d}x A(x) B(x) = \int \mathrm{d}x B(x) * A(x) \,, \tag{20}$$

a special case of which is given by

$$\int (A * B * C) = \int (B * C * A) = \int (C * A * B) \,. \tag{21}$$

First, we vary the gauge field $A$ to see the variation of the action (1). We follow the convention of keeping the field to be varied at the extreme left. Then we get

$$\frac{\delta S}{\delta A_\sigma(y)} = \int \mathrm{d}^4x \left[ -\mathrm{Tr}\frac{\delta F_{\mu\nu}(x)}{\delta A_\sigma(y)} * F^{\mu\nu}(x) + \frac{\delta}{\delta A_\sigma(y)}(-g\bar{\psi} * \gamma^\nu A_\nu * \psi) \right] ,$$

where, in the first integral, (20) has been used. Further simplifications are done by using the cyclicity property (21) in the second integral of the above expression. We obtain

$$\frac{\delta S}{\delta A_\sigma(y)} = \int \mathrm{d}^4x \Big[ -\mathrm{Tr}\frac{\delta}{\delta A_\sigma(y)} \times (\partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu]_*) * F^{\mu\nu}(x) + \frac{\delta}{\delta A_\sigma(y)} \left(g A_\nu * \psi_j \gamma^\nu_{ij} * \bar{\psi}_i\right) \Big] .$$

The extra negative sign in the second integral is due to the flip of two grassmannian variables. Finally, using the fact that $F^{\mu\nu}$ is antisymmetric in its indices, we obtain

$$\begin{aligned} \frac{\delta S}{\delta A_\sigma(y)} &= \int \mathrm{d}^4x \Big[ -2\mathrm{Tr}\frac{\delta}{\delta A_\sigma(y)} \times (\partial_\mu A_\nu + \mathrm{i}g A_\mu * A_\nu) * F^{\mu\nu}(x) + \frac{\delta A_\nu(x)}{\delta A_\sigma(y)} * \left(g\psi_j \gamma^\nu_{ij} * \bar{\psi}_i\right) \Big] \\ &= \int \mathrm{d}^4x \Big[ 2\mathrm{Tr}\frac{\delta A_\nu(x)}{\delta A_\sigma(y)} * (\partial_\mu F^{\mu\nu}(x) + \mathrm{i}g A_\mu * F^{\mu\nu} - \mathrm{i}g F^{\mu\nu} * A_\mu) + \frac{\delta A_\nu(x)}{\delta A_\sigma(y)} * \left(g\psi_j \gamma^\nu_{ij} * \bar{\psi}_i\right) \Big] . \end{aligned} \tag{22}$$

The invariance of the action together with (17) now leads to the equation of motion for the gauge field,

$$\partial_\mu F^{\mu\nu} + \mathrm{i}g[A_\mu, F^{\mu\nu}]_* + j^\nu = 0 \,, \tag{23}$$

where $j^\nu$ is the fermionic current,

$$j^\nu = g\psi_j (\gamma^\nu)_{ij} * \bar{\psi}_i \,. \tag{24}$$

The variation of the matter field $\bar{\psi}$ in the action (1) yields

$$\begin{aligned} \frac{\delta S}{\delta \bar{\psi}(y)} &= \int \mathrm{d}^4x \frac{\delta}{\delta \bar{\psi}(y)} \left(\bar{\psi} * i\gamma^\mu \partial_\mu \psi - g\bar{\psi} * \gamma^\mu A_\mu * \psi - m\bar{\psi} * \psi\right) \\ &= \int \mathrm{d}^4x \frac{\delta \bar{\psi}(x)}{\delta \bar{\psi}(y)} * (\mathrm{i}\gamma^\mu \partial_\mu \psi - g\gamma^\mu A_\mu * \psi - m\psi) \,, \end{aligned} \tag{25}$$

which gives the equation of motion

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - g\gamma^\mu A_\mu * \psi - m\psi = 0 \,. \tag{26}$$

Similarly, for the other matter field we get the equation of motion:

$$\mathrm{i}\partial_\mu \bar{\psi} \gamma^\mu + g\bar{\psi} * \gamma^\mu A_\mu + m\bar{\psi} = 0 \,. \tag{27}$$

Operating by $\partial_\nu$ on (23) we get a current conservation law [6]:

$$\partial_\nu J^\nu = 0; J^\nu \equiv \mathrm{i}g[A_\mu, F^{\mu\nu}]_* + j^\nu \,. \tag{28}$$

This can be explicitly verified from the definition of $J^\nu$, as follows:

$$\begin{aligned}\partial_\nu J^\nu &= \mathrm{i}g[\partial_\nu A_\mu, F^{\mu\nu}]_* + \mathrm{i}g[A_\mu, \partial_\nu F^{\mu\nu}]_* + \partial_\nu j^\nu \\ &= -\frac{1}{2}\mathrm{i}g[\partial_\mu A_\nu - \partial_\nu A_\mu, F^{\mu\nu}]_* \\ &\quad + \mathrm{i}g\,[A_\mu, \mathrm{i}g[A_\nu, F^{\nu\mu}]_* + j^\mu]_* + \partial_\nu j^\nu \\ &= -\frac{1}{2}\mathrm{i}g[F_{\mu\nu}, F^{\mu\nu}]_* \\ &\quad + \frac{(\mathrm{i}g)^2}{2}\left[[A_\mu, A_\nu]_*, F^{\mu\nu}\right]_* - (\mathrm{i}g)^2\left[A_\mu, [A_\nu, F^{\mu\nu}]_*\right]_* \\ &\quad + \mathrm{i}g[A_\mu, j^\mu]_* + \partial_\nu j^\nu \,.\end{aligned}$$

Since the first term on the right hand side vanishes trivially, we write the above expression as

$$\begin{aligned}\partial_\nu J^\nu = \frac{(\mathrm{i}g)^2}{2}\Big(&[[A_\mu, A_\nu]_*, F^{\mu\nu}]_* - [A_\mu, [A_\nu, F^{\mu\nu}]_*]_* \\ &+ [A_\nu, [A_\mu, F^{\mu\nu}]_*]_*\Big) + \mathrm{i}g[A_\mu, j^\mu]_* + \partial_\nu j^\nu \,.\end{aligned}$$

The term in the parentheses vanishes from the Jacobi identity, and we obtain

$$\partial_\nu J^\nu = \mathrm{i}g[A_\mu, j^\mu]_* + \partial_\nu j^\nu. \tag{29}$$

(Star) multiplying (26) by $-\mathrm{i}g\bar{\psi}$ from the right and (27) by $-\mathrm{i}g\psi$ from the left, and then adding those two equations, we find

$$\partial_\nu J^\nu = \mathrm{i}g[A_\mu, j^\mu]_* + \partial_\nu j^\nu = 0\,, \tag{30}$$

where the definition (24) of $j^\mu$ has been used. It is also possible to obtain the current defined in (28) from (1) by using a Noether-like procedure [5]. If we make the following "global" transformation on the gauge and matter fields:

$$\delta A_\mu(x) = \mathrm{i}g[\omega(x), A_\mu(x)]_*\,, \tag{31}$$

$$\delta\psi(x) = -\mathrm{i}g\omega(x) * \psi(x)\,, \tag{32}$$

$$\delta\bar{\psi}(x) = \mathrm{i}g\bar{\psi}(x) * \omega(x)\,, \tag{33}$$

and set $\omega(x)$ constant at the end of the calculation, the conserved current (28) follows from (1).

As has been stressed [5] the conservation law (28) is compatible with both types of gauge symmetry (6) (with the Leibniz rule (7)) and (8) (with the Leibniz rule (9)). One finds, for instance, $\partial_\mu(\delta_* J^\mu) = \partial_\mu(\delta_\eta J^\mu) = 0$. It is clear that the conservation law is unable to provide any distinction between the two types of gauge transformations. This is not surprising, since this conservation law is an on-shell symmetry that is quite distinct from the gauge symmetry that is an off-shell symmetry. So in the next two sections we study the gauge (both star gauge and twisted gauge) symmetry of the system where on-shell considerations are discarded. The difference between the star gauge and twisted gauge symmetries thereby is made manifest.

## 3 Analysis for star gauge transformation

As already stated, the presence of gauge symmetry is characterised by an identity that is called the gauge identity. In this section we first discuss a general formalism to connect this identity with the gauge generator that eventually leads to the gauge transformations. Next we use this method for the particular model (1) to find the gauge generators from which the star deformed gauge transformations (6) are systematically obtained.

Consider the general form of the action on noncommutative space:[2]

$$S = \int \mathrm{d}tL = \int \mathrm{d}^4x\mathcal{L}\left(q_\alpha(\mathbf{x},t), \partial_i q_\alpha(\mathbf{x},t), \partial_t q_\alpha(\mathbf{x},t)\right)\,, \tag{34}$$

where $\alpha$ denotes the number of fields. An arbitrary variation of this action can be written as

$$\delta S = -\int \mathrm{d}^4x\delta q^\alpha(\mathbf{x},t) * L_\alpha(\mathbf{x},t)\,, \tag{35}$$

where the vanishing of the Euler derivative $L$ yields the equations of motion. Now, if we vary the field $q^\alpha$ in the following way:

$$\delta q^\alpha(\mathbf{x},t) = \sum_{s=0}^{n}(-1)^s \int \mathrm{d}^3\mathbf{z}\frac{\partial^s\eta^b(\mathbf{z},t)}{\partial t^s} * \rho^{\alpha b}_{(s)}(x,z) \tag{36}$$

with $\eta$ and $\rho$ being the parameter and generator, respectively, of the transformation, the variation of the action can be written by (35) as

$$\begin{aligned}\delta S = &-\int \mathrm{d}^4x \int \mathrm{d}^3\mathbf{z}\eta^b(\mathbf{z},t) * \rho^{\alpha b}_{(0)}(x,z) * L_\alpha(\mathbf{x},t) \\ &- \int \mathrm{d}^4x \sum_{s=1}^{n}(-1)^s \int \mathrm{d}^3\mathbf{z}\frac{\partial}{\partial t}\left(\frac{\partial^{s-1}\eta^b(\mathbf{z},t)}{\partial t^{s-1}}\right) \\ &\qquad * \rho^{\alpha b}_{(s)}(x,z) * L_\alpha(\mathbf{x},t) \\ = &-\int \mathrm{d}^4x \int \mathrm{d}^3\mathbf{z}\eta^b(\mathbf{z},t) * \rho^{\alpha b}_{(0)}(x,z) * L_\alpha(\mathbf{x},t) \\ &- \int \mathrm{d}^4x \sum_{s=1}^{n}(-1)^{s-1} \int \mathrm{d}^3\mathbf{z}\frac{\partial^{s-1}\eta^b(\mathbf{z},t)}{\partial t^{s-1}} \\ &\qquad * \frac{\partial}{\partial t}\left(\rho^{\alpha b}_{(s)}(x,z) * L_\alpha(\mathbf{x},t)\right) \\ = &-\int \mathrm{d}^4z\eta^b(\mathbf{z},t) * \left(\int \mathrm{d}^3\mathbf{x}\rho^{\alpha b}_{(0)}(x,z) * L_\alpha(\mathbf{x},t)\right) \\ &- \int \mathrm{d}^4z\eta^b(\mathbf{z},t) * \left(\int \mathrm{d}^3\mathbf{x}\frac{\partial}{\partial t}\left(\rho^{\alpha b}_{(1)}(x,z) * L_\alpha(\mathbf{x},t)\right)\right) \\ &- \cdots\end{aligned} \tag{37}$$

Equation (37) is written in the compact form

$$\delta S = -\int \mathrm{d}^4z\eta^a(\mathbf{z},t) * \Lambda^a(\mathbf{z},t) \tag{38}$$

[2] Here we adopt the notation $x$ for the four vector $x^\mu = (\mathbf{x}, t)$.

where[3]

$$\Lambda^a(\mathbf{z},t)=\left[\sum_{s=0}^{n}\int \mathrm{d}^3\mathbf{x}\frac{\partial^s}{\partial t^s}\left(\rho^{\alpha a}_{(s)}(x,z)*L_\alpha(\mathbf{x},t)\right)\right]. \quad (39)$$

If the action is invariant ($\delta S=0$), then it implies

$$\Lambda^a(\mathbf{z},t)=0\,. \quad (40)$$

The last equality must be identically valid without use of any equation of motion. It is called the gauge identity. Equation (36) defines the gauge transformation of the fields with $\rho$ being the generator. Furthermore, the gauge identity involves the generator and Euler derivatives in a specific fashion given by (39).

There are now two ways to apply this general formulation to a specific gauge model. Starting from knowledge of the gauge transformations $\delta q$ (obtained, for instance, by inspection) it should be possible to compute the generators $\rho$ by using (36). Then the explicit structure for the gauge identity follows from (39). Alternatively, one starts from the gauge identity (obtained as shown below), inverts the process, finally generating the gauge transformations. Here we adopt the second approach for the model (1).

The first step to obtain the gauge identity is to derive the Euler derivatives. This is simply done by considering an arbitrary variation of the action (1), expressed in terms of the variations of the basic fields,

$$\delta S=-\int \mathrm{d}^4x\delta A^a_\mu * L^{\mu a}+\delta\psi_i * L_i+\delta\bar{\psi}_i * L'_i \quad (41)$$

where the Euler derivatives $L^a_\mu$, $L_i$ and $L'_i$ are given by

$$L^{\mu a}=-\left(\mathcal{D}_\sigma * F^{\sigma\mu}\right)^a-g\psi_j(\gamma^\mu T^a)_{ij}*\bar{\psi}_i \quad (42)$$
$$L_i=-\mathrm{i}\partial_\mu\bar{\psi}_j(\gamma^\mu)_{ji}-g\bar{\psi}_j*(\gamma^\mu A^a_\mu T^a)_{ji}-m\bar{\psi}_i \quad (43)$$
$$L'_i=-\mathrm{i}(\gamma^\mu)_{ij}\partial_\mu\psi_j+g(\gamma^\mu A^a_\mu T^a)_{ij}*\psi_j+m\psi_i\,. \quad (44)$$

Here the covariant derivative $\mathcal{D}$ is defined in the adjoint representation,

$$\mathcal{D}_\mu *\xi=\partial_\mu\xi+\mathrm{i}g[A_\mu,\xi]_*\,, \quad (45)$$
$$(\mathcal{D}_\mu*\xi)^a=\partial_\mu\xi^a-\frac{g}{2}f^{abc}\left\{A^b_\mu,\xi^c\right\}_*+\mathrm{i}\frac{g}{2}d^{abc}\left[A^b_\mu,\xi^c\right]_*\,, \quad (46)$$

where we have used (12) and (16). We now define a quantity $\Lambda$, involving the various Euler derivatives of the system as follows:

$$\Lambda^a=-\left(\mathcal{D}^\mu * L_\mu\right)^a-\mathrm{i}gT^a_{ij}\psi_j*L_i-\mathrm{i}gT^a_{ji}L'_i*\bar{\psi}_j\,. \quad (47)$$

Exploiting the definitions of the covariant derivative (45) and the Euler derivatives ((42), (43) and (44)), the above expression, by an explicit calculation, is found to be zero, i.e. it vanishes identically without using any equations of motion,

$$\Lambda^a=-\left(\mathcal{D}^\mu * L_\mu\right)^a-\mathrm{i}gT^a_{ij}\psi_j*L_i-\mathrm{i}gT^a_{ji}L'_i*\bar{\psi}_j=0\,. \quad (48)$$

The above relation is the cherished gauge identity for the model (1). It is important to note that the structure of $\Lambda^a$ in (47) is similar to the general form (39) in the sense that it involves the appropriate Euler derivatives. By a comparison of the two, the generators $\rho$ are obtained. To this end, let us now write (39) in a convenient way that is more suitable for our particular model,

$$\Lambda^a(\mathbf{z},t)=\sum_s\int \mathrm{d}^3\mathbf{x}\frac{\partial^s}{\partial t^s}\left(\rho^{b\mu a}_{(s)}(x,z)*L^b_\mu(\mathbf{x},t)\right)+\sum_s\int \mathrm{d}^3\mathbf{x}\frac{\partial^s}{\partial t^s}\left(\phi^a_i(x,z)*L_i(\mathbf{x},t)+\phi'^a_i(x,z)*L'_i(\mathbf{x},t)\right). \quad (49)$$

The values of the generators $\rho$, $\phi$ and $\phi'$ are obtained by comparing (47) and (49). Since the calculations involve some subtlety due to the noncommutative nature of the coordinates, a couple of intermediate steps are presented here. The contribution coming from the zeroth component of the gauge field Euler derivative $L_\mu$ can be written by (48) as

$$\Lambda^a|_{L_0}=-\left(\mathcal{D}^0 * L_0\right)^a=-\frac{g}{2}f^{abc}\left\{L^b_0,A^{0c}\right\}_*+\mathrm{i}\frac{g}{2}d^{abc}\left[L^b_0,A^{0c}\right]_*-\frac{\partial}{\partial t}L^a_0\,. \quad (50)$$

We write the above equation in the following form:

$$\begin{aligned}\Lambda^a|_{L_0}(\mathbf{z},t)=&-\frac{g}{2}f^{abc}\int \mathrm{d}^3\mathbf{x}\left(L^b_0(x)*A^{0c}(x)+A^{0c}(x)*L^b_0(x)\right)*\delta^3(\mathbf{x}-\mathbf{z})\\&-\frac{\mathrm{i}}{2}d^{abc}\int \mathrm{d}^3\mathbf{x}\left(L^b_0(x)*A^{0c}(x)-A^{0c}(x)*L^b_0(x)\right)*\delta^3(\mathbf{x}-\mathbf{z})\\&-\int \mathrm{d}^3\mathbf{x}\frac{\partial}{\partial t}L^a_0(x)*\delta^3(\mathbf{x}-\mathbf{z})\,,\end{aligned} \quad (51)$$

where we have used the property (20). Furthermore, exploiting the cyclicity property of the star product (21), (51) is further simplified, so as to bring the Euler derivative at an extreme end,

$$\begin{aligned}\Lambda^a|_{L_0}(\mathbf{z},t)=&-\int \mathrm{d}^3\mathbf{x}\frac{g}{2}\left(f^{abc}\left\{\delta^3(\mathbf{x}-\mathbf{z}),A^{0c}(x)\right\}_*+\mathrm{i}d^{abc}\left[\delta^3(\mathbf{x}-\mathbf{z}),A^{0c}(x)\right]_*\right)*L^b_0(x)\\&-\int \mathrm{d}^3\mathbf{x}\delta^{ab}\delta^3(\mathbf{x}-\mathbf{z})*\frac{\partial}{\partial t}L^b_0(x)\,.\end{aligned} \quad (52)$$

The same contribution coming from (49) can be written as

$$\Lambda^a|_{L_0}(\mathbf{z},t)=\sum_s\int \mathrm{d}^3\mathbf{x}\frac{\partial^s}{\partial t^s}\left(\rho^{b0a}_{(s)}(x,z)*L^b_0(\mathbf{x},t)\right). \quad (53)$$

[3] Equations (38) and (39) are the star deformed version of the commutative space results given, for instance, in [7, 12].

Only $s=0,1$ contribute, so that the above equation simplifies to

$$\Lambda^a|_{L_0}(\mathbf{z},t) = \int \mathrm{d}^3\mathbf{x}\left(\rho^{b0a}_{(0)}(x,z) * L^b_0(\mathbf{x},t) + \rho^{b0a}_{(1)}(x,z) * \frac{\partial}{\partial t} L^b_0(\mathbf{x},t)\right). \quad (54)$$

Comparing (52) and (54), we obtain

$$\rho^{b0a}_{(0)}(x,z) = -\frac{g}{2} f^{abc}\left\{\delta^3(\mathbf{x}-\mathbf{z}), A^c_0(x)\right\}_* - \mathrm{i}\frac{g}{2} d^{abc}\left[\delta^3(\mathbf{x}-\mathbf{z}), A^c_0(x)\right]_* \quad (55)$$

$$\rho^{b0a}_{(1)}(x,z) = -\delta^{ab}\delta^3(\mathbf{x}-\mathbf{z}). \quad (56)$$

The other components of the gauge generator can be obtained in a similar way. Here we give the full expressions of these components, which will be useful in finding the gauge transformations of the various fields.

$$\rho^{bia}_{(0)}(x,z) = -\delta^{ab}\partial^{i\mathbf{z}}\delta^3(\mathbf{x}-\mathbf{z}) - \frac{g}{2} f^{abc}\left\{\delta^3(\mathbf{x}-\mathbf{z}), A^{ic}(x)\right\}_* - \mathrm{i}\frac{g}{2} d^{abc}\left[\delta^3(\mathbf{x}-\mathbf{z}), A^{ic}(x)\right]_*, \quad (57)$$

$$\phi^a_{i(0)}(x,z) = -\mathrm{i}gT^a_{ij}\delta^3(\mathbf{x}-\mathbf{z}) * \psi_j(x), \quad (58)$$

$$\phi'^a_{i(0)}(x,z) = -\mathrm{i}gT^a_{ji}\bar{\psi}_j(x) * \delta^3(\mathbf{x}-\mathbf{z}). \quad (59)$$

Let us next consider the gauge transformations. From (36) we write the gauge transformation equation for the zeroth component of the gauge field

$$\delta A^{0a}(\mathbf{x},t) = \sum_s (-1)^s \int \mathrm{d}^3\mathbf{z}\frac{\partial^s \eta^b(\mathbf{z},t)}{\partial t^s} * \rho^{a0b}_{(s)}(x,z) = \int \mathrm{d}^3\mathbf{z}\left(\eta^b(\mathbf{z},t) * \rho^{a0b}_{(0)}(x,z) - \frac{\partial \eta^b(\mathbf{z},t)}{\partial t} * \rho^{a0b}_{(1)}(x,z)\right). \quad (60)$$

Exploiting the identity [8, 9]

$$A(x) * \delta(x-z) = \delta(x-z) * A(z) \quad (61)$$

and interchanging $a, b$, the generator (55) is recast to

$$\rho^{a0b}_{(0)}(x,z) = \frac{g}{2} f^{abc}\left\{\delta^3(\mathbf{x}-\mathbf{z}), A^c_0(z)\right\}_* + \mathrm{i}\frac{g}{2} d^{abc}\left[\delta^3(\mathbf{x}-\mathbf{z}), A^c_0(z)\right]_*. \quad (62)$$

Use of (62) and (56) along with the identities (20) and (21) in (60) implies that

$$\delta A^{0a} = \partial^0\eta^a - \frac{g}{2} f^{abc}\left\{A^{0b}, \eta^c\right\}_* + \mathrm{i}\frac{g}{2} d^{abc}\left[A^{0b}, \eta^c\right]_* = \left(\mathcal{D}^0 * \eta\right)^a, \quad (63)$$

where the operator $\mathcal{D}$ had already been defined in (45). In a similar way, using the expression (59), we can get the space component of the gauge transformation of the $A^\mu$ field:

$$\delta A^{ia} = \partial^i\eta^a - \frac{g}{2} f^{abc}\left\{A^{ib}, \eta^c\right\}_* + \mathrm{i}\frac{g}{2} d^{abc}\left[A^{ib}, \eta^c\right]_* = \left(\mathcal{D}^i * \eta\right)^a. \quad (64)$$

Combining the two results (63) and (64), we get the following star covariant gauge transformation rule for the gauge field

$$\delta A^{\mu a} = \left(\mathcal{D}^\mu * \eta\right)^a. \quad (65)$$

The same process leads to the star gauge transformation relations of the matter fields:

$$\delta\psi_i(x) = -\mathrm{i}g\eta^a(x) * T^a_{ij}\psi_j(x) \quad (66)$$

$$\delta\bar{\psi}_i(x) = \mathrm{i}gT^a_{ji}\bar{\psi}_j(x) * \eta^a(x). \quad (67)$$

Thus the star gauge transformations of all the fields have been systematically obtained. These transformations ((65), (66) and (67)) are the results previously stated in Sect. 2, (6), under which the action (1) is invariant. Also, the generators $\rho$ are mapped with the gauge identity $\Lambda^a$ (48) by the relation (39). If we set $\theta = 0$, then these just correspond to the usual commutative space results for Yang–Mills theory in the presence of matter [7]. This implies that, as it occurs for the gauge transformations, the mapping (39) is also a star deformation of the usual undeformed (commutative space) map.

Let us now mention a technical point. In obtaining the gauge transformations – say (63) from (60) – use is made of identities like (20) and (21), which are strictly valid over the whole four dimensional space-time. Since (60) involves only the space integral, manipulations based on these identities imply only space–space noncommutativity. This is quite reminiscent of the Hamiltonian formulation of gauge symmetries [9], where $\theta^{0i} = 0$ from the beginning.

We conclude this section by providing a simple consistency check. We show that the variation of the action (1) is indeed expressed in the form (38), where $\Lambda^a$ is given by (47). Starting from (41) and using the explicit structures of the variations derived in (65), (66) and (67)), we obtain

$$\delta S = -\int \mathrm{d}^4x(\mathcal{D}^\mu * \eta)^a * L^a_\mu + (-\mathrm{i}g\eta^a * T^a_{ij}\psi_j) * L_i + (\mathrm{i}gT^a_{ji}\bar{\psi}_j * \eta^a) * L'_i = -\int \mathrm{d}^4x\eta^a * \left((-\mathcal{D}^\mu * L_\mu)^a - \mathrm{i}gT^a_{ij}\psi_j * L_i - \mathrm{i}gT^a_{ji}L'_i * \bar{\psi}_j\right). \quad (68)$$

The expression star multiplied with the gauge parameter $\eta^a$ is precisely $\Lambda^a$ given by (47), conforming to the general form (38). This completes the consistency check.

## 4 Analysis for twisted gauge transformation

For simplicity we take the pure gauge theory,

$$S = -\frac{1}{2}\int \mathrm{d}^4x \mathrm{Tr}\left(F_{\mu\nu}(x) * F^{\mu\nu}(x)\right) , \qquad (69)$$

where the field strength tensor was defined in (3). Now the gauge field transforms in the undeformed way,

$$\delta_\eta A_\mu = \partial_\mu \eta + \mathrm{i}g\left[A_\mu, \eta\right] . \qquad (70)$$

Using the deformed coproduct rule (9) and the gauge transformation (70), the variation of the (star) product of the gauge fields is also seen to be undeformed,

$$\delta_\eta\left(A_\mu * A_\nu\right) = \partial_\mu\eta A_\nu + A_\mu\partial_\nu\eta - \mathrm{i}g\left[\eta, \left(A_\mu * A_\nu\right)\right] . \qquad (71)$$

From the above result, the gauge transformation of the field strength tensor is now computed and found to be

$$\begin{aligned}\delta_\eta F_{\mu\nu} &= \partial_\mu\delta_\eta A_\nu - \partial_\nu\delta_\eta A_\nu + \mathrm{i}g\delta_\eta\left[A_\mu, A_\nu\right]_* &(72)\\ &= \partial_\mu\left(\partial_\nu\eta + \mathrm{i}g\left[A_\nu,\eta\right]\right) - \partial_\nu\left(\partial_\mu\eta + \mathrm{i}g\left[A_\mu,\eta\right]\right)\\ &\quad + \mathrm{i}g\left(\left[\partial_\mu\eta, A_\nu\right] + \left[A_\mu, \partial_\nu\eta\right] - \mathrm{i}g\left[\eta,\left[A_\mu, A_\nu\right]_*\right]\right) &(73)\\ &= -\mathrm{i}g\left[\eta, F_{\mu\nu}\right] . &(74)\end{aligned}$$

Likewise, one finds that the expression $F^{\mu\nu} * F_{\mu\nu}$ transforms as

$$\delta_\eta\left(F^{\mu\nu} * F_{\mu\nu}\right) = -\mathrm{i}g\left[\eta, F^{\mu\nu} * F_{\mu\nu}\right] . \qquad (75)$$

Both $F_{\mu\nu}$ and $F_{\mu\nu} * F^{\mu\nu}$ have the usual (undeformed) transformation properties. Thus the action (69) is invariant under the gauge transformation (70) and the deformed coproduct rule (9).

There is another way of interpreting the gauge invariance that makes contact with the gauge identity. Making a gauge variation of the action (69) and taking into account the twisted coproduct rule (9), we get

$$\begin{aligned}\delta_\eta S &= -\frac{1}{2}\int \mathrm{d}^4x \mathrm{Tr}\delta_\eta\left(F_{\mu\nu} * F^{\mu\nu}\right) &(76)\\ &= -\frac{1}{2}\int \mathrm{d}^4x \Big[\mathrm{Tr}\Big(\delta_\eta F_{\mu\nu} * F^{\mu\nu} + F_{\mu\nu} * \delta_\eta F^{\mu\nu}\\ &\quad - \frac{\mathrm{i}}{2}\theta^{\mu_1\nu_1}\left(\delta_{\partial_{\mu_1}\eta}F_{\mu\nu} * \partial_{\nu_1}F^{\mu\nu} + \partial_{\mu_1}F_{\mu\nu} * \delta_{\partial_{\nu_1}\eta}F^{\mu\nu}\right)\\ &\quad - \frac{1}{8}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\Big(\delta_{\partial_{\mu_1}\partial_{\mu_2}\eta}F_{\mu\nu} * \partial_{\nu_1}\partial_{\nu_2}F^{\mu\nu}\\ &\quad + \partial_{\mu_1}\partial_{\mu_2}F_{\mu\nu} * \delta_{\partial_{\nu_1}\partial_{\nu_2}\eta}F^{\mu\nu}\Big) + \cdots\Big)\Big] . &(77)\end{aligned}$$

Now using the result (73) each term of (77) can be computed separately. For example, let us concentrate on the first term. Using the identity (20) and the trace condition (17), we write the first term

$$\begin{aligned}\delta_\eta S|_{\text{1st term}} &= -\frac{1}{4}\int \mathrm{d}^4x\left(\delta_\eta F^{\mu\nu a} * F^a_{\mu\nu} + F^{\mu\nu a} * \delta_\eta F^a_{\mu\nu}\right) &(78)\\ &= -\frac{1}{2}\int \mathrm{d}^4x\delta_\eta F^{\mu\nu a}F^a_{\mu\nu} . &(79)\end{aligned}$$

Making use of (73) and dropping the surface terms, the above expression is found to be

$$\begin{aligned}\delta_\eta S|_{\text{1st term}} = -\int \mathrm{d}^4x\eta^a\big(&-\partial^\mu\partial^\nu F_{\mu\nu} - \mathrm{i}g\partial^\mu\left[A^\nu, F_{\mu\nu}\right]\\ &- \mathrm{i}g\left[A^\mu, \partial^\nu F_{\mu\nu}\right] + g^2\left[A^\mu * A^\nu, F_{\mu\nu}\right]\big)^a .\end{aligned} \qquad (80)$$

The second term of (77) is identically zero due to the antisymmetric nature of $\theta^{\mu\nu}$. We write it

$$\begin{aligned}\delta_\eta S|_{\text{2nd term}} &= -\frac{1}{2}\int \mathrm{d}^4x\eta^a\frac{\mathrm{i}}{2}\theta^{\mu_1\nu_1}\left(-\mathrm{i}g\left\{\partial_{\mu_1}F^{\mu\nu}, \partial_{\nu_1}F_{\mu\nu}\right\}\right)^a &(81)\\ &= -\int \mathrm{d}^4x\eta^a\frac{\mathrm{i}}{2}\theta^{\mu_1\nu_1}\big(-\mathrm{i}g\left\{\partial_{\mu_1}\partial^\mu A^\nu, \partial_{\nu_1}F_{\mu\nu}\right\}\\ &\quad + g^2\left\{\partial_{\mu_1}\left(A^\mu * A^\nu\right), \partial_{\nu_1}F_{\mu\nu}\right\}\big)^a &(82)\\ &= -\int \mathrm{d}^4x\eta^a\frac{\mathrm{i}}{2}\theta^{\mu_1\nu_1}\big(-\mathrm{i}g\partial^\mu\left\{\partial_{\mu_1}A^\nu, \partial_{\nu_1}F_{\mu\nu}\right\}\\ &\quad - \mathrm{i}g\left\{\partial_{\mu_1}A^\mu, \partial_{\nu_1}\partial^\nu F_{\mu\nu}\right\}\\ &\quad + g^2\left\{\partial_{\mu_1}\left(A^\mu * A^\nu\right), \partial_{\nu_1}F_{\mu\nu}\right\}\big)^a . &(83)\end{aligned}$$

The third term is written as

$$\begin{aligned}\delta_\eta S|_{\text{3rd term}} = -\int \mathrm{d}^4x\left(\partial^{\mu_1}\partial^{\mu_2}\eta^a\right)\frac{1}{2}\left(\frac{\mathrm{i}}{2}\right)^2\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\\ \times\big(-\partial^\mu\partial^\nu\partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\\ - \mathrm{i}g\partial^\mu\left[A^\nu, \partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\right]\\ - \mathrm{i}g\left[A^\mu, \partial^\nu\partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\right]\\ + g^2\left[A^\mu * A^\nu, \partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\right]\big)^a .\end{aligned} \qquad (84)$$

Using the antisymmetry of $\theta^{\mu\nu}$ and dropping the various surface terms, we write the above expression as

$$\begin{aligned}\delta_\eta S|_{\text{3rd term}} = -\int \mathrm{d}^4x\eta^a\frac{1}{2}\left(\frac{\mathrm{i}}{2}\right)^2\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\\ \times\big(-\mathrm{i}g\partial^\mu\left[\partial^{\mu_1}\partial^{\mu_2}A^\nu, \partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\right]\\ - \mathrm{i}g\left[\partial^{\mu_1}\partial^{\mu_2}A^\mu, \partial^\nu\partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\right]\\ + g^2\left[\partial^{\mu_1}\partial^{\mu_2}\left(A^\mu * A^\nu\right), \partial^{\nu_1}\partial^{\nu_2}F_{\mu\nu}\right]\big)^a .\end{aligned} \qquad (85)$$

The other terms can be obtained in a similar manner. Combining all these terms, we finally get

$$\begin{aligned}\delta_\eta S &= -\int \mathrm{d}^4x\eta^a\big(-\partial^\mu\partial^\nu F_{\mu\nu} - \mathrm{i}g\partial^\mu\left[A^\nu, F_{\mu\nu}\right]_*\\ &\quad - \mathrm{i}g\left[A^\mu, \partial^\nu F_{\mu\nu}\right]_* + g^2\left[A^\mu * A^\nu, F_{\mu\nu}\right]_*\big)^a &(86)\\ &= -\int \mathrm{d}^4x\eta^a\Lambda^a , &(87)\end{aligned}$$

where

$$\Lambda^a = -\left(\mathcal{D}^\mu * L_\mu\right)^a = -\left(\mathcal{D}^\mu * \mathcal{D}^\sigma * F_{\sigma\mu}\right)^a , \qquad (88)$$

which vanishes identically. Note that this is exactly the same as the expression in the gauge identity (48) without the fermionic fields. This proves the invariance of the action.

Let us now repeat the analysis of the previous section with appropriate modifications. Since the gauge transformations are undeformed, the gauge generators are expected to have the same form as in the commutative space. To see this, note that the gauge variation of the zeroth component of the $A_\mu$ field, following from (70), can be written as

$$\delta_\eta A_0^a(z) = \partial_0\eta^a(z) - gf^{abc}A_0^b(z)\eta^c(z) = g\int \mathrm{d}^3\mathbf{z} f^{abc}A_0^c\eta^b\delta^3(\mathbf{x}-\mathbf{z}) + \int \mathrm{d}^3\mathbf{z}\delta^{ab}\delta^3(\mathbf{x}-\mathbf{z})\frac{\partial}{\partial t}\eta^b . \quad (89)$$

Clearly the above result can be expressed in our standard form (36),

$$\delta_\eta A_0^a(z) = \sum_s(-1)^s\int \mathrm{d}^3\mathbf{z}\frac{\partial^s\eta^b(\mathbf{z},t)}{\partial t^s}\rho^{a0b}_{(s)}(x,z) = \int \mathrm{d}^3\mathbf{z}\eta^b(\mathbf{z},t)\rho^{a0b}_{(0)}(x,z) - \int \mathrm{d}^3\mathbf{z}\frac{\partial\eta^b(\mathbf{z},t)}{\partial t}\rho^{a0b}_{(1)}(x,z)\,, \quad (90)$$

where

$$\rho^{a0b}_{(0)}(x,z) = gf^{abc}A_0^c\delta^3(\mathbf{x}-\mathbf{z}) \quad (91)$$
$$\rho^{a0b}_{(1)}(x,z) = -\delta^{ab}\delta^3(\mathbf{x}-\mathbf{z}) \quad (92)$$

is the gauge generator. Similarly

$$\delta_\eta A_i^a(z) = \partial_i\eta^a(z) - gf^{abc}A_i^b(z)\eta^c(z) \quad (93)$$

is written in the form

$$\delta_\eta A_i^a(z) = \sum_s(-1)^s\int \mathrm{d}^3\mathbf{z}\frac{\partial^s\eta^b(\mathbf{z},t)}{\partial t^s}\rho^{aib}_{(s)}(x,z) \quad (94)$$

for the value

$$\rho^{aib}_{(0)}(x,z) = -\delta^{ab}\partial^{i\mathbf{z}}\delta^3(\mathbf{x}-\mathbf{z}) + gf^{abc}A_i^c\delta^3(\mathbf{x}-\mathbf{z})\,. \quad (95)$$

No star products appear in the gauge generators $\rho$, and their structure is similar to the undeformed commutative space expressions. To identify the difference (both from the commutative space results and the star deformed results) it is essential to look at the gauge identity and its connection with the corresponding gauge generator.

Now as already implied in (88), we have a gauge identity for this system, exactly similar to the previous case,

$$\Lambda^a = -\left(\mathcal{D}^\mu * L_\mu\right)^a = 0\,, \quad (96)$$

where $L_\mu$ is the Euler derivative defined in (88). The Euler–Lagrange equation of motion is given by

$$\mathcal{D}^\sigma * F_{\sigma\mu} = 0\,. \quad (97)$$

The gauge identity and the Euler derivatives are mapped by the relation

$$\Lambda^a(\mathbf{z},t) = \sum_{s=0}^{n}\int \mathrm{d}^3\mathbf{x}\frac{\partial^s}{\partial t^s}\left(\rho'^{b\mu a}_{(s)}(x,z)L^b_\mu(\mathbf{x},t)\right)\,, \quad (98)$$

where the values of $\rho'^{b\mu a}_{(0)}(x,z)$ and $\rho'^{b\mu a}_{(1)}(x,z)$ are equal to those of $\rho^{b\mu a}_{(0)}$ and $\rho^{b\mu a}_{(1)}$ of the previous example, given in (55), (56) and (59). This happens since the Euler derivatives and the gauge identity are identical to those discussed in the previous section. However, here $\rho'$ is not the generator; rather $\rho$ is (see (91), (92) and (95)). Consequently $\rho'$ has to be expressed in terms of $\rho$. To do this, we rewrite (55) under the identification $\rho=\rho'$ as

$$\rho'^{b0a}_{(0)}(x,z) = -\frac{g}{2}f^{abc}\left\{\delta^3(\mathbf{x}-\mathbf{z}),A_0^c(x)\right\}_* - \mathrm{i}\frac{g}{2}d^{abc}\left[\delta^3(\mathbf{x}-\mathbf{z}),A_0^c(x)\right]_*\,. \quad (99)$$

Now making use of the definition of the star product (5), the above expression is written in the following way:

$$\rho'^{b0a}_{(0)}(x,z) = -gf^{abc}A_0^c\delta^3(\mathbf{x}-\mathbf{z}) - g\sum_{n=1}^{\infty}\left(\frac{\mathrm{i}}{2}\right)^n\frac{\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}}{n!} \times\left[\left(\frac{f^{abc}}{2}+\mathrm{i}\frac{d^{abc}}{2}\right)\partial_{\mu_1}\cdots\partial_{\mu_n}\delta^3(\mathbf{x}-\mathbf{z})\partial_{\nu_1}\cdots\partial_{\nu_n}A^{0c}(x)\right. \left.\left(+\frac{f^{abc}}{2}-\mathrm{i}\frac{d^{abc}}{2}\right)\partial_{\mu_1}\cdots\partial_{\mu_n}A^{0c}(x)\partial_{\nu_1}\cdots\partial_{\nu_n}\delta^3(\mathbf{x}-\mathbf{z})\right]\,. \quad (100)$$

Note that the $\theta$ independent term is nothing but the gauge generator $\rho^{b0a}_{(0)}$ given in (91). Similarly calculating the other components $\rho'^{bia}_{(0)}$ and $\rho'^{b0a}_{(1)}$ from (56) and (59), we obtain

$$\rho'^{b\mu a}_{(0)}(x,z) = \rho^{b\mu a}_{(0)}(x,z) - g\sum_{n=1}^{\infty}\left(\frac{\mathrm{i}}{2}\right)^n\frac{\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}}{n!} \times\left[\left(\frac{f^{abc}}{2}+\mathrm{i}\frac{d^{abc}}{2}\right)\partial_{\mu_1}\cdots\partial_{\mu_n}\delta^3(\mathbf{x}-\mathbf{z})\partial_{\nu_1}\cdots\partial_{\nu_n}A^{\mu c}(x)\right. \left.\left(+\frac{f^{abc}}{2}-\mathrm{i}\frac{d^{abc}}{2}\right)\partial_{\mu_1}\cdots\partial_{\mu_n}A^{\mu c}(x)\partial_{\nu_1}\cdots\partial_{\nu_n}\delta^3(\mathbf{x}-\mathbf{z})\right] \quad (101)$$

$$\rho'^{b0a}_{(1)}(x,z) = \rho^{b0a}_{(1)}(x,z). \quad (102)$$

We conclude that, although the generator remains undeformed, the relation mapping the gauge identity with the generator is neither the commutative space result nor its star deformation as found in the other approach. Rather, it is twisted from the undeformed result. The additional twisted terms are explicitly given in (101).

## 5 Conclusions

Gauge symmetries on canonically deformed coordinate spaces have been considered. Both possibilities (namely, deformed gauge transformations keeping the standard Leibniz rule intact or undeformed gauge transformations with a twisted Leibniz rule) were analysed within a common framework. Explicit structures of the gauge generators were obtained in either case. The connection of these generators with the gauge identity, which must exist whenever there is a gauge symmetry, was also established. In the former case, this connection was a star deformation of the commutative space result. In the latter case, on the other hand, the commutative space result was appropriately twisted. It is quite remarkable that these fundamental properties of the gauge symmetries (i.e. the occurrence of the gauge identity and its connection with the corresponding generator through the Euler derivatives) were found in the noncommutative theory, adopting either of the two interpretations. This suggests that deformed gauge theories have properties similar to what we desire for physics, at least as far as gauge symmetries are concerned. All results obtained here reduce to the usual commutative space expressions in the limit of vanishing $\theta$.

## References


1. P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml, J. Wess, arXiv:hep-th/0603024
2. J. Wess, arXiv:hep-th/0608135
3. D.V. Vassilevich, Mod. Phys. Lett. A **21**, 1279 (2006) [arXiv:hep-th/0602185]
4. M. Chaichian, A. Tureanu, Phys. Lett. B **637**, 199 (2006) [arXiv:hep-th/0604025]
5. L. Alvarez-Gaume, F. Meyer, M.A. Vazquez-Mozo, Nucl. Phys. B **753**, 92 (2006) [arXiv:hep-th/0605113]
6. J. Wess, arXiv:hep-th/0607251
7. D.M. Gitman, I.V. Tyutin, Quantization of Fields with Constraints (Springer, Berlin Heidelberg, 1990)
8. R. Amorim, F.A. Farias, Phys. Rev. D **65**, 065 009 (2002) [arXiv:hep-th/0109146]
9. R. Banerjee, Phys. Rev. D **67**, 105 002 (2003) [arXiv:hep-th/0210259]
10. L. Bonora, M. Salizzoni, Phys. Lett. B **504**, 80 (2001) [arXiv:hep- th/0011088]
11. A. Armoni, Nucl. Phys. B **593**, 229 (2001) [arXiv:hep-th/0005208]
12. A. Shirzad, J. Phys. A **31**, 2747 (1998)